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Acceleration-induced radiative excitation of ground-state atoms

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Abstract
We use elementary time-dependent perturbation theory, referred wholly to an inertial (laboratory) frame, to determine the probability that a semi-realistically modelled atom is promoted from the ground to an excited state, with the emission of a photon, when its nucleus is constrained to follow a classically prescribed trajectory including a finite interval of (arbitrary) acceleration between asymptotically uniform initial and final motions. In the formal limit where the proper acceleration α is constant and lasts forever, we verify the Unruh effect, namely that the atom then behaves as if it had been exposed to black-body radiation at temperature $T_U = \hbar\alpha/2\pi c k_B$. The point is that in virtue of its simplicity our formalism is reasonably adaptable, and its predictions free of objections like those often and rightly based on the unrealizable nature of strictly constant α considered directly rather than as a limit.

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1. Introduction

It is often asserted that 'an accelerated observer perceives the vacuum as a thermal bath of photons'. Consider an atom moving through the vacuum (i.e. through the ground state of the quantized Maxwell field) at constant proper acceleration of magnitude α, for all τ ranging from $-\infty$ to $+\infty$ (τ denotes the proper time). A more precise version is that in its instantaneous rest frame the atom will experience the vacuum as if the atom were at rest, and interacting with a heat bath at temperature T_U given by

$$k_B T_U = \alpha\hbar/2\pi c, \tag{1}$$

the subscript U standing for 'Unruh' (Davies 1975, Unruh 1976). Early papers on the subject and reviews include Unruh and Wald (1984), Takagi (1986), Brout *et al* (1995). The formal connection between constant acceleration and isometry in Lorentz space-time is made by Haag (1993).

One consequence of the statement is that an atom accelerated in vacuum should undergo spontaneous excitation, as if by absorbing a heat bath photon; such photons will also affect the rates of downward transitions. However, the precise physical significance of the formally elegant results found in the literature cited above is often somewhat obscure: witness the current debate on the subject as traced through, say, Fulling and Unruh (2004), Narozhny *et al* (2004), Scully *et al* (2003, 2004), Hu and Roura (2004), and Hu *et al* (2004). In our view, confusion sometimes arises because (a) much of the discussion focuses on what an *accelerated* observer perceives, whereas measurements are generally made in inertial frames; and because (b) constant accelerations of infinite duration are unrealizable.

We intend to study the case of a two-particle (conventionally 'nucleus' and 'electron') bound state following a realizable trajectory, and interacting with a scalar radiation field. The calculation of transition probabilities will feature nothing more exotic than simple time-dependent perturbation theory. Such a model has been presented, with the present problem in mind, by Barton and Calogeracos (2005), referred to as BCI. We start with a field-theoretic model of a bound state, because point-particle models acting as substitutes for an atom coupled to a quantized field cannot be guaranteed not to oversimplify. The acceleration is formally implemented by an explicitly time-dependent Hamiltonian constraint acting on the nucleus. This is sketched in section 2; more detail may be found in BCI.

We restrict ourselves to asymptotically inertial linear trajectories (which are physically realizable). We consider an atom that starts at time (or equivalently proper time) $-\infty$ in its ground state i, with no photons present. We are interested in the probability that later on it has been promoted to an excited state f, with the emission of a photon having wave-vector $\mathbf{k}$. Let $c_{f\mathbf{k}}$ be the corresponding transition amplitude. The transition probability is given by

$$P_f(\tau) = \int \mathrm{d}^3k\, P_{f\mathbf{k}}(\tau), \qquad P_{f\mathbf{k}}(\tau) = |c_{f\mathbf{k}}(\tau)|^2. \tag{2}$$

In section 3 we present expression (21) for $c_{f\mathbf{k}}(\tau)$ (to lowest order in the coupling constant), in terms of the velocity $\mathbf{B}(\tau)$ characterizing an asymptotically inertial trajectory, without commitment to any particular $\mathbf{B}(\tau)$. The general expression (21) is then evaluated for the special case of a nucleus moving along the z axis, with its velocity B such that initially (for $-\infty < \tau < -\tau_0$) $B = -B_0$, finally (for $\tau_0 < \tau < \infty$) $B = B_0$, and at the intermediate acceleration stage

$$B = c \tanh(\alpha\tau/c). \tag{3}$$

It is convenient to introduce the scaled proper time x, and a crucial parameter x_0:

$$x \equiv \alpha\tau/c, \qquad x_0 \equiv \alpha\tau_0/c. \tag{4}$$

In the limit $x_0 \to \infty$ the trajectory becomes the strictly hyperbolic Unruh trajectory: for brevity we shall call this the 'Unruh scenario'. Expression (27) for the transition amplitude features the factor $L = L_{if} + L_a$, where L_{if} given by (34) stems from the initial and final stages, and L_a given by (41) and (42) from the acceleration stage.

In section 4 we determine the limiting form of the amplitudes for $x_0 \to \infty$ in terms of Bessel functions K with pure imaginary order and real argument. We also define a transition rate W as transition probability per unit of proper time and show that in the limit where the duration of the acceleration goes to infinity the limiting form of the rate is *as if* the atom were at rest and interacting with a reservoir of photons at temperature T_U (1).

2. A relativistic Hamiltonian for a bound state with the nucleus following a prescribed trajectory

We start from a Hamiltonian describing a massless scalar radiation field interacting with the nucleus considered as a spin 0 point particle having no internal degrees of freedom, and with a Dirac electron. The nucleus follows a prescribed relativistic trajectory $\mathbf{R}(t)$, with $\mathbf{B}(t) \equiv \mathrm{d}\mathbf{R}/\mathrm{d}t$ the corresponding velocity. (The subscripts $\parallel, \perp$ shall specify vector components parallel and perpendicular to $\mathbf{B}(t)$.) Since the position coordinate of the nucleus is externally prescribed (hence a c number in the quantum theory), the nuclear Hamiltonian is just minus the nuclear Lagrangian (i.e. the Routhian). Thus the appropriate Hamiltonian has the form

$$H = H_{\mathrm{rad}} + (M + Zg\phi(\mathbf{R}))\sqrt{1 - B^2} + H_{\mathrm{electron}} \tag{5}$$

where H_{rad} is the usual massless-scalar-field Hamiltonian, and

$$H_{\mathrm{electron}} = \alpha \cdot \mathbf{p} + \beta m + \beta g\phi(\mathbf{r})$$

features a mass-like coupling of the electron to the field.

We subject H to a sequence of canonical transformations aiming at a more convenient Hamiltonian (see BCI for details):

(i) A time-dependent translation which amounts to moving to the nucleus rest-frame, so that $\mathbf{R} = 0$ for all t.
(ii) A so-called *passive-source* transformation, removing the nucleus–field coupling in favour (initially) of a 4-scalar electron–nucleus potential

$$V_B(r) = -\frac{(g^2/4\pi)}{\sqrt{r_\parallel^2/(1 - B^2) + r_\perp^2}}. \tag{6}$$

(iii) A scale transformation on the components of $\mathbf{r}$ and $\mathbf{p}$, related to the Lorentz contraction.
(iv) A boost.
(v) A phase transformation[3] whose action on the zero-order stationary states $|n\rangle$ of the atom depends explicitly on the corresponding eigenvalue ε_n.

These transformations eventually yield an effective Hamiltonian and an effective state vector,

$$\mathrm{i}\frac{\partial}{\partial t}|\rangle_{\mathrm{eff}} = H_{\mathrm{eff}}|\rangle_{\mathrm{eff}}, \tag{7}$$

$$H_{\mathrm{eff}} = \left\{ H_{\mathrm{rad}} - \mathbf{B} \cdot \mathbf{P}_{\mathrm{rad}} + \sqrt{1 - B^2}\sum_n |n\rangle\varepsilon_n\langle n| \right\} + H_{\mathrm{int}} + (\text{noninertial terms}) \tag{8}$$

where

$$H_{\mathrm{int}} = \sqrt{1 - B^2}\sum_n\sum_q |n\rangle\langle n|g\beta \exp(-\mathrm{i}\varepsilon_{nq}Br_\parallel)\phi\left(r_\parallel\sqrt{1 - B^2}, \mathbf{r}_\perp\right)|q\rangle\langle q| \tag{9}$$

with $\varepsilon_{nq} \equiv \varepsilon_n - \varepsilon_q$,

$$\phi(\mathbf{x}) = \int \mathrm{d}^3k\left\{a_{\mathbf{k}}\phi_{\mathbf{k}}(\mathbf{x}) + a_{\mathbf{k}}^\dagger\phi_{\mathbf{k}}^*(\mathbf{x})\right\}, \qquad \phi_{\mathbf{k}}(\mathbf{x}) \equiv \frac{1}{4\pi^{3/2}k^{1/2}}\exp(\mathrm{i}\mathbf{k} \cdot \mathbf{x}), \tag{10}$$

$$\left[a_{\mathbf{k}}, a_{\mathbf{k}'}^+\right] = \delta(\mathbf{k} - \mathbf{k}'), \tag{11}$$

[3] BCI called this a gauge instead of a phase transformation.

while H_{rad} and the field momentum $\mathbf{P}_{\mathrm{rad}}$ assume the form

$$(H_{\mathrm{rad}}, \mathbf{P}_{\mathrm{rad}}) = \sum_{\mathbf{k}} (k, \mathbf{k}) a_{\mathbf{k}}^{\dagger} a_{\mathbf{k}}. \tag{12}$$

A few explanations are in order:

(a) Equation (8) features the (zero-order) atomic eigenvalues $\varepsilon_n \sqrt{1 - B^2}$, with ε_n the eigenvalues stemming from (6) for an atom at rest $\mathbf{B} = \mathbf{0}$,

$$\{\alpha \cdot \mathbf{p} + \beta m + V_0(r)\}|n\rangle = \varepsilon_n |n\rangle. \tag{13}$$

Equation (13) is tackled by Greiner *et al* (1985). We shall consider only weak coupling, taking

$$g^2/4\pi \approx 1/137 \ll 1. \tag{14}$$

This entails light binding in the sense that the internal motion becomes nonrelativistic, with the same bound-state energies and wavefunctions as for the true nonrelativistic hydrogen atom:

$$\varepsilon_n \simeq m - (g^2/4\pi)^2 m/2n^2, \qquad n = 1, 2, 3, \ldots; \tag{15}$$

$$\langle \mathbf{r}|n\rangle \equiv \psi_n(\mathbf{r}) \propto \exp(-r/na), \qquad a = 1/m(g^2/4\pi), \tag{16}$$

where a is the Bohr radius.

(b) The time dependence in transformation (i) leads to the crucial term $\mathbf{B} \cdot \mathbf{P}_{\mathrm{rad}}$ in (8). The noninertial terms in (8) stem from the explicit time dependence of the canonical transformations (ii)–(v) on $\mathbf{B}(t)$. We do not attempt to calculate them since such an endeavour would open a whole new field of inquiry. Observe that noninertial contributions have never been treated in the Unruh literature. In fact their calculation requires a dynamical model which is provided herein but is absent from most works in the subject.

(c) We focus on the perturbative calculation of excitation probabilities to leading order and we thus drop zero-point contributions.

The square root in the eigenvalues $\varepsilon_n \sqrt{1 - B^2}$ correctly takes into account time dilation for a moving atom. Also (BCI, section 5), at constant $\mathbf{B}$, the several explicit factors $\sqrt{1 - B^2}$ conspire with the phase factor inside the matrix element to ensure, without any further (e.g. without any multipolar) approximations, that excited states decay at the properly time-dilated rates, i.e. with decay constants

$$\Gamma(\mathbf{B}) = \Gamma(0)\sqrt{1 - B^2}. \tag{17}$$

In fact the conspiracy is so elaborate, and the result appears so unheralded and so late, that one develops considerable confidence that, within its obvious limits of applicability, H_{eff} does indeed deliver results conformable to special relativity[4]. Such optimism is further encouraged by detailed examination (not reported here) of the Lorentz-transformation properties of the total energy and momentum operators, which because of the constraint is not immediately obvious how to identify.

Finally, it is worth stressing the evident technical distinction between calculations needed (i) for the spontaneous decay of excited atomic states (e.g. $2pm$ to $1s$, say f to i, plus photon) that are allowed even inertially (at constant $\mathbf{B}$), and (ii) excitations (e.g. $1s$ to $2pm$, say i to f,

[4] See for instance Boussiakou *et al* (2002), Cresser and Barnett (2003), and Horsley and Babiker (2005) for other ways to time-dilate uniformly moving atoms coupled to the Maxwell field, starting from the standard nonrelativistic two-body Schrödinger equation, and without introducing constraints. For an approach based on the Bethe–Salpeter equation, see Järvinen (2005).

plus photon) that are possible only for varying $\mathbf{B}$, with the necessary energy $(\varepsilon_{fi}+k)$ supplied by the agency enforcing the constraint on $\mathbf{R}(t)$. It may be profitable to recall the point stressed in detail in BCI. The formal constraint represents a force acting on the nucleon *only.* The Coulomb field forces the electron to follow suit and the electron–photon interaction gives rise to the type (ii) inertially forbidden transitions that interest us here.

3. Perturbation theory

3.1. Generalities

We wish to calculate the transition amplitudes $c_{f\mathbf{k}}(t)$ for an atomic up transition from $i = 1s$ to $f = 2pm$ (in standard spectroscopic notation), accompanied by the emission of a single photon $\mathbf{k}$ into the initial vacuum. The present subsection admits arbitrary trajectories $\mathbf{B}(t)$.

The amplitudes will be determined by the text-book first-order time-dependent perturbation theory, treating the curly bracket in (8) as the zero-order Hamiltonian, and H_{int} of (9) as the perturbation. Without loss of generality, we measure phases from $t = 0 = \tau$. Then, in an obvious notation, the standard perturbative Ansatz, starting bare at $t = -\infty$, i.e. with $c_{q\mathbf{k}}(-\infty) = 0$, reads

$$|\psi;t\rangle = |i\rangle|0\rangle \exp\left\{-\mathrm{i}\varepsilon_i\int_0^t \mathrm{d}t'\sqrt{1-B^2(t')}\right\}$$
$$+\sum_q\int \mathrm{d}^3k|q\rangle|\mathbf{k}\rangle \exp\left\{-\mathrm{i}\int_0^t \mathrm{d}t'[k-\mathbf{B}(t')\cdot\mathbf{k}+\varepsilon_q\sqrt{1-B^2(t')}]\right\}c_{q\mathbf{k}}(t),$$

where $|0\rangle$ is the vacuum (no-photon) state and $|\mathbf{k}\rangle$ the state with one photon having wave-vector $\mathbf{k}$. This yields the first-order solution

$$c_{f\mathbf{k}}(\infty) = -\mathrm{i}\int_{-\infty}^{\infty}\mathrm{d}t\,F\langle f|H_{\text{int}}(t)|i\rangle \exp\left\{\mathrm{i}\int_0^t \mathrm{d}t'[\varepsilon_{fi}\sqrt{1-B^2(t')}+k-\mathbf{B}(t')\cdot\mathbf{k}]\right\}, \tag{18}$$

where the auxiliary function $F(\tau)$, of the form $F = \exp(-\lambda|\tau|)$, switches off the interaction adiabatically and covariantly as $|\tau| \to \pm\infty$. The switching factor is needed in principle to make sense of the outer integrals in (18) over the initial and final uniform velocity stages, but disappears from the end-results upon taking the limit $\lambda \to 0$ afterwards.

The matrix element in (18) is

$$\frac{g}{4\pi^{3/2}k^{1/2}}\sqrt{1-B^2}\langle f|\beta\exp\{-\mathrm{i}[\varepsilon_{fi}\mathbf{B}\cdot\mathbf{r}+k_\parallel r_\parallel\sqrt{1-B^2}+\mathbf{k}\cdot\mathbf{r}_\perp]\}|i\rangle. \tag{19}$$

Hence

$$\begin{aligned} c_{f\mathbf{k}}(\infty) = -\frac{\mathrm{i}g}{4\pi^{3/2}k^{1/2}}&\int_{-\infty}^{\infty}\mathrm{d}t\,F(\tau)\sqrt{1-B^2(t)} \\ &\times\exp\left\{\mathrm{i}\int_0^t \mathrm{d}t'[\varepsilon_{fi}\sqrt{1-B^2(t')}+k-\mathbf{B}(t')\cdot\mathbf{k})]\right\} \\ &\times\langle f|\beta\exp\{-\mathrm{i}[\varepsilon_{fi}B(t)r_\parallel+k_\parallel r_\parallel\sqrt{1-B^2(t)}+\mathbf{k}\cdot\mathbf{r}_\perp]\}|i\rangle, \end{aligned} \tag{20}$$

and expressed in terms of proper time τ

$$\begin{aligned} c_{f\mathbf{k}}(\infty) = -\frac{\mathrm{i}g}{4\pi^{3/2}k^{1/2}}&\int_{-\infty}^{\infty}\mathrm{d}\tau\,F(\tau)\exp\left\{\mathrm{i}\int_0^\tau \mathrm{d}\tau'\left[\varepsilon_{fi}+\frac{k-\mathbf{B}(\tau')\cdot\mathbf{k}}{\sqrt{1-B^2(\tau')}}\right]\right\} \\ &\times\langle f|\beta\exp\{-\mathrm{i}[\varepsilon_{fi}B(\tau)r_\parallel+k_\parallel r_\parallel\sqrt{1-B^2(\tau)}+\mathbf{k}\cdot\mathbf{r}_\perp]\}|i\rangle. \end{aligned} \tag{21}$$

3.2. The acceleration scenario, range of parameters and matrix elements

From here on we consider only the rectilinear trajectory of the type described just before (3). We introduce the scaled variables

$$h \equiv c\varepsilon_{fi}/\hbar\alpha, \qquad \kappa \equiv \mathbf{k}c^2/\alpha. \tag{22}$$

(h ought not to be confused with Planck's constant.) The atomic unit of acceleration is (see (15) and the second of (16) for the Bohr radius)

$$\alpha_{\text{atomic}} = g^2/4\pi a^2 m = (g^2/4\pi)^2 c^2/a \simeq 0.90 \times 10^{25}\text{cm s}^{-2}, \tag{23}$$

whereas laboratory accelerations are unlikely to exceed those of order 10^8 cm s^{-2} achievable by the ultracentrifuge. Thus we confine our attention to

$$h \gg 1. \tag{24}$$

The result (57) and a close look at the properties of the modified Bessel function featuring in the latter also entail

$$\kappa/h \sim \mathcal{O}(1). \tag{25}$$

The atomic matrix elements that enter (21) when $i = 1s$ and $f = 2pm, m = 0, \pm 1$ are calculated in Barton and Calogeracos (2008), hereafter referred to as BCII. The exact expressions are complicated functions of B and thereby of τ, and using them is well beyond the scope of the present report. They simplify considerably in the internally-nonretarded approximation (equivalent in our case to the standard dipole approximation), but even then it is only those for $m = \pm 1$ that reduce to constants (independent of B). We confine ourselves to such transitions, for which the dipole approximation yields (after performing the radial and angular integrations)

$$\langle 2p, \pm 1|\ldots|1s\rangle \quad = \quad Mak_{\pm}, \qquad M = \pm \mathrm{i}128/243, \tag{26}$$

in natural units $\hbar = c = 1$.

3.3. The structure of the transition amplitude $c_{f\mathbf{k}}$

In the scenario and with the matrix elements just discussed, equations (21) and (2) lead to

$$c_{f\mathbf{k}} = \frac{gaM}{4\pi^{3/2}} \cdot \frac{k_{\pm}}{k^{1/2}} L, \tag{27}$$

$$P_f = \frac{g^2a^2\,|M|^2}{16\pi^3} \int \mathrm{d}^3k \frac{k_{\perp}^2}{k} |L|^2, \tag{28}$$

where

$$L = \left\{ \int_{-\infty}^{-\tau_0} + \int_{-\tau_0}^{\tau_0} + \int_{\tau_0}^{\infty} \right\} \mathrm{d}\tau \exp(-\mathrm{i}I(\tau)) = L_i + L_a + L_f \tag{29}$$

requires the phases

$$I(\tau) \equiv \int_0^{\tau} \mathrm{d}\tau' \left\{ \varepsilon_{fi} + \frac{k - B(\tau')k_3}{\sqrt{1 - B^2(\tau')}} \right\}. \tag{30}$$

It is straightforward to evaluate $I(\tau)$ both in the initial and final stages $|\tau| > \tau_0$, and in the acceleration stage $|\tau| < \tau_0$, defined in the few lines preceding (3). Details are given in BCII.

3.4. The amplitudes $L_{i,f}$

These amplitudes are easy to evaluate, by virtue of the adiabatic switching factors F, and because (30) shows that the phases $I_{i,f}$ are linear in x. We define the polar angle ϑ of κ with respect to $\mathbf{B}$:

$$\kappa_3 = \kappa\cos\vartheta, \qquad \kappa_\perp = \kappa\sin\vartheta. \tag{31}$$

Eventually we find

$$L_i = \frac{(\mathrm{i}/\alpha)}{[h+\kappa\cosh(x_0)+\kappa_3\sinh(x_0)]}\exp\{\mathrm{i}[hx_0+\kappa\sinh(x_0)+\kappa_3(\cosh(x_0)-1))]\}, \tag{32}$$

$$L_f = \frac{(-\mathrm{i}/\alpha)}{[h+\kappa\cosh(x_0)-\kappa_3\sinh(x_0)]}\exp\{\mathrm{i}[-hx_0-\kappa\sinh(x_0)+\kappa_3(\cosh(x_0)-1))]\}. \tag{33}$$

For nonzero x_0 the two amplitudes combine into

$$L_{if} \equiv L_i + L_f = \frac{(-2\mathrm{i}/\alpha)\exp\{\mathrm{i}\kappa_3(\cosh x_0-1)\}}{[(h+\kappa\cosh x_0)^2-\kappa_3^2\sinh^2 x_0]} \times\{-\mathrm{i}(h+\kappa\cosh x_0)\sin(\kappa\sinh x_0+hx_0)+\kappa_3\sinh x_0\cos(\kappa\sinh x_0+hx_0)\}. \tag{34}$$

As $x_0\to\infty$, clearly L_i, L_f, and therefore L_{if} vanish like $\exp(-x_0)$. Specifically

$$|L_{if}|^2 \to \frac{16\exp(-2x_0)}{\alpha^2\kappa_\perp^4}\{\kappa^2\sin^2[\kappa\sinh x_0+hx_0]+\kappa_3^2\cos^2[\kappa\sinh x_0+hx_0]\}. \tag{35}$$

3.5. The amplitude L_a

If we write

$$L_a = \frac{\exp(-\mathrm{i}\kappa_3)}{\alpha}Q, \tag{36}$$

it eventually follows from (30) that

$$Q \equiv \int_{-x_0}^{x_0}\mathrm{d}x\,\exp\{-\mathrm{i}[hx+\kappa\sinh x-\kappa_3\cosh x]\}. \tag{37}$$

After some changes of variable and the definition (useful in what follows)

$$x_3 \equiv \log\left[\frac{\kappa-\kappa_3}{\kappa_\perp}\right] = \log[\tan(\vartheta/2)] \tag{38}$$

we obtain

$$Q = \exp(\mathrm{i}hx_3)\int_{\xi_1}^{\xi_2}\frac{\mathrm{d}\xi}{\xi}\exp\left\{-\mathrm{i}\left[h\log\xi+\frac{\kappa_\perp}{2}\left(\xi-\frac{1}{\xi}\right)\right]\right\}, \tag{39}$$

where $\xi_{2,1} = \tan(\vartheta/2)\exp(\pm x_0)$. After a further change of variables we obtain

$$Q = \exp\{\mathrm{i}hx_3\}J(h,\kappa_\perp), \tag{40}$$

where

$$J \equiv \int_{u_1}^{u_2}\mathrm{d}u\,\exp\{-\mathrm{i}[hu+\kappa_\perp\sinh(u)]\}, \quad u_2 = x_0+x_3, \qquad u_1 = -x_0+x_3. \tag{41}$$

Thus the end-result reads

$$L_a = \exp\{\mathrm{i}[hx_3-\kappa_3]\}J/\alpha. \tag{42}$$

4. The limit $x_0 \to \infty$

4.1. The amplitude

In this regime one need consider only L_a, since L_{if} vanishes like $\exp(-x_0)$ (see the remark preceding (35)). Hence we can write

$$\lim_{x_0\to\infty} L = \lim_{x_0\to\infty} L_a \equiv L_\infty. \tag{43}$$

The limit $x_0 \to \infty$ entails $u_2 \to \infty$ and $u_1 \to -\infty$. In virtue of the increasingly fast oscillation of the integrand, the integral J remains convergent, so that

$$\begin{aligned} J_\infty \equiv \lim_{x_0\to\infty} J &= \int_{-\infty}^{\infty} \mathrm{d}u \exp\{-\mathrm{i}[hu + \kappa_\perp \sinh(u)]\} \\ &= 2\int_0^\infty \mathrm{d}u\{\cos[hu]\cos[\kappa_\perp \sinh(u)] - \sin[hu]\sin[\kappa_\perp \sinh(u)]\}. \end{aligned} \tag{44}$$

The relations (Abramowitz and Stegun (1965), equations 9.6.22 and 9.6.24)

$$\begin{aligned} K_\nu(x) &= \sec(\nu\pi/2)\int_0^\infty \mathrm{d}s \cos(x\sinh s)\cosh(\nu s) \\ &= \csc(\nu\pi/2)\int_0^\infty \mathrm{d}s \sin(x\sinh s)\sinh(\nu s), \qquad (|\mathrm{Re}\,\nu| < 1, x > 0), \end{aligned} \tag{45}$$

$$K_\nu(x) = \int_0^\infty \mathrm{d}s \exp(-x\cosh s)\cosh(\nu s), \qquad (|\arg x| < \pi/2) \tag{46}$$

then lead to

$$J_\infty = 2\{\cos(\mathrm{i}h\pi/2) + \mathrm{i}\sin(\mathrm{i}h\pi/2)\}K_{ih}(\kappa_\perp) = 2\exp(-h\pi/2])K_{ih}(\kappa_\perp), \tag{47}$$

$$L_\infty = \frac{2}{\alpha}\exp\{\mathrm{i}[hx_3 - \kappa_3]\}\exp(-\pi h/2)K_{ih}(\kappa_\perp), \qquad x_3 \equiv \log[\tan(\vartheta/2)]. \tag{48}$$

According to (24) we work in the regime where h and $\kappa_\perp$ are comparably large; however, because the order is pure imaginary, one cannot rely on the relatively simple asymptotic formulae for Bessel functions K found in the literature. Note also that K is real, and that for transversely emitted photons (48) reduces to $L_\infty(\vartheta = \pi/2) = (2/\alpha)\exp(-\pi h/2)K_{ih}(\kappa)$.

4.2. The transition rate

We restrict ourselves to large durations of the acceleration and accordingly use the limiting form of L (43). We substitute in expression (28) for the transition probability and write

$$P_f(\tau) = \frac{g^2a^2|M|^2}{16\pi^3}\int \mathrm{d}^3k\frac{k_\perp^2}{k}|L_\infty(\tau)|^2 = \frac{g^2a^2|M|^2\alpha^4}{16\pi^3}\int \mathrm{d}^3\kappa\frac{\kappa_\perp^2}{\kappa}|L_\infty(\tau)|^2. \tag{49}$$

We define the transition rate

$$W = \lim_{\tau\to\infty}\frac{1}{2}\frac{\mathrm{d}P_f(\tau)}{\mathrm{d}\tau} = \lim_{x_0\to\infty}\frac{\alpha}{2}\frac{\mathrm{d}P_f}{\mathrm{d}x_0} \tag{50}$$

(where the factor $1/2$ stems from the fact that the duration of the acceleration is 2τ). Relation (18) leads to

$$W = \frac{g^2a^2|M|^2\alpha^5}{16\pi^3}\lim_{x_0\to\infty}\int \mathrm{d}^3\kappa\frac{\kappa_\perp^2}{\kappa}\mathrm{Re}\left\{L_\infty^*\frac{\mathrm{d}L_\infty}{\mathrm{d}x_0}\right\}. \tag{51}$$

The quantity $\frac{\mathrm{d}L_\infty}{\mathrm{d}x_0}$ may be calculated via (41) and (42). Thus

$$W = \frac{g^2a^2|M|^2\alpha^3}{4\pi^3}\,\mathrm{e}^{-\pi h/2}\lim_{x_0\to\infty}\int \mathrm{d}^3\kappa\,\frac{\kappa_\perp^2}{\kappa}K_{ih}(\kappa_\perp)\cos[hx_3+\kappa_\perp\cosh(x_0)\sinh(x_3)] \times\cos[hx_0+\kappa_\perp\sinh(x_0)\cosh(x_3)]. \tag{52}$$

We split the above integration into Cartesian components, recall definition (38) of x_3 and rearrange the integrand:

$$W = \frac{g^2a^2|M|^2\alpha^3}{4\pi^3}\mathrm{e}^{-\pi h/2}\lim_{x_0\to\infty}\int \mathrm{d}\kappa_\perp\kappa_\perp^3K_{ih}(\kappa_\perp)I \tag{53}$$

$$I\equiv I^+ + I^- \tag{54}$$

$$I^- = \int_{-\infty}^{\infty}\frac{\mathrm{d}\kappa_3}{\kappa}\cos\{h[x_0-\log\kappa_\perp+\log(\kappa-\kappa_3)]+\kappa_\perp\sinh[x_0-\log\kappa_\perp+\log(\kappa-\kappa_3)]\} \tag{55}$$

$$I^+ = I^-(\kappa_3\to-\kappa_3). \tag{56}$$

We make successive changes of variable from κ_3 to $t = \sqrt{\kappa_\perp^2+\kappa_3^2}-\kappa_3$ and from t to $y=\log t + x_0 - \log\kappa_\perp$. Then W simplifies considerably:

$$W = \frac{g^2a^2|M|^2\alpha^3}{2\pi^2}\,\mathrm{e}^{-\pi h}\int \mathrm{d}\kappa_\perp\kappa_\perp^3K_{ih}^2(\kappa_\perp). \tag{57}$$

We make use of the remarkable integral (Prudnikov *et al* 1990)

$$\int_0^\infty \mathrm{d}x\,x^3K_{ih}^2(x) = \frac{1}{3}|\Gamma(2+ih)|^2 = \frac{1}{3}(1+h^2)\frac{\pi h}{\sinh(\pi h)} \tag{58}$$

to obtain the final expression for the transition rate

$$W = \frac{g^2}{3\pi}a^2|M|^2\varepsilon_{fi}^3\frac{(1+1/h^2)}{\mathrm{e}^{2\pi h}-1}. \tag{59}$$

Notice the Planckian factor, with $2\pi h = 2\pi\varepsilon_{fi}/\alpha = \varepsilon_{fi}/k_BT_U$ featuring the Unruh temperature. The $1/h^2$ in the numerator of (59) is a correction to the Unruh result (see also Marzlin and Audretsch (1998)).

5. Conclusion

We considered a bound state consisting of a scalar nucleus plus a Dirac electron, based on a relativistic field-theoretic model, and constrained the nucleus to move along a prescribed asymptotically inertial trajectory which at intermediate times corresponds to uniform acceleration. In the limit $\tau\to\infty$ (the duration of the acceleration stage tending to infinity) our trajectory generates the Unruh scenario. We calculated the transition rate W as defined in (50), for the atom to be found in an excited final state at proper time $\tau\to\infty$ if it starts in its ground state. The rate indeed coincides with the Unruh result modulo a correction term in the numerator. Comparing the present approach to more conventional ones we observe the following:

(i) Following Unruh's work there has been near unanimous agreement in the literature concerning the excitation rate (59). However there has been much discussion concerning the nature (even the existence) of the emitted radiation. Our approach is firmly based on a laboratory frame treatment and the photon states describe actual photons labelled by the lab momentum.

(ii) The present model is based on a modestly realistic coupling between particles and field rather than on artificial point couplings.

(iii) Realistic trajectories involve accelerations of finite duration and the present approach delivers $W(\tau)$ equally easily. In fact, as will be reported in BCII moderate acceleration durations $x_0 \sim \mathcal{O}(1)$ result in transition probabilities higher by some orders of magnitude than the results quoted for $x_0 \to \infty$. This is a rather more promising prospect if one contemplates observing such effects in the laboratory.

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